Fractional Tikhonov regularization revisited

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Abstract

It is well known and observed again and again that classical Tikhonov regularization for ill-posed inverse problems leads to rather smooth reconstructions, often smoother than one would like. In an attempt to remedy this, fractional Tikhonov regularization was introduced. In fact, two methods with the name were suggested. The method of [1] aims at solving the normal equation

$$((A^*A)^{(\alpha+1)/2} + \mu I)x = (A^*A)^{(\alpha-1)/2}A^*b^\delta,$$

where the parameter $0 < \alpha < 1$ is used to reduce the smoothing compared to standard Tikhonov regularization which corresponds to $\alpha = 1$. The second method, proposed in [2], aim at solving the linear system

$$(A^*A + \mu I)^\alpha x = (A^*A)^{\alpha-1}A^*b^\delta$$

where α plays the same role and $\alpha = 1$ yields standard Tikhonov regularization. Both methods were compared in [3], and several papers transferred the concept of fractional regularization to approaches other than Tikhonov regularization.

A main point that, to the best of the authors knowledge, has not been answered anywhere up to now is the choice of the fractional parameter α. In this talk, we give an answer to that based on a new way of interpreting regularization introduced in [4]. We also discuss why, from the point of regularization theory, fractional regularization proposes no significant benefit over classical Tikhonov regularization and even fractional regularization with $\alpha > 1$, i.e., amplified smoothing of the solutions. A main ingredient to this is understanding the role of the optimal choice of the regularization parameter μ, which leads to all methods yielding, up to constants, the same upper and lower bound on the reconstruction error.

Keywords: Tikhonov regularization, inverse problem, fractional regularization

References

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