Primal and dual descent regularization algorithms for imaging

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Abstract

Let us consider the functional equation $Ax = y$ characterized by an ill-posed linear operator $A : X \longrightarrow Y$ between two Banach spaces X and Y. In this talk, we study on-step iterative gradient methods to minimize the residual functional $\Phi(x) = \frac{1}{p}\|Ax - y\|_Y^p$, with $p > 1$. It is well-known that gradient methods can be considered as implicit regularization algorithm if combined with an early-stopping criterion to prevent over-fitting of the noise.

We prove that gradient methods in Banach spaces can be fully explained and understood in the context of proximal operator theory, with appropriate norm (for primal schemes) or Bregman (for dual schemes) distances as proximity measure, which shows a deep connection between iterative regularisation and convex optimisation. In particular, we consider the following proximal iterative method

$$x_{k+1} = \arg\min_{x \in X} \left\{ B_r^X(x, x_k) + \alpha_k \langle \partial\Phi(x_k), x\rangle \right\}$$

where $\alpha_k > 0$ is a proper step length, and B_r^X is the Bregman distance with respect to the functional $\frac{1}{r}\|\cdot\|_X^r$ with $r > 1$, that is

$$B_r^X(x, x_k) = \frac{1}{r}\|x\|_X^r - \Big(\frac{1}{r}\|x_k\|_X^r + \langle \partial\big(\frac{1}{r}\|x_k\|_X^r\big), x - x_k\rangle\Big),$$

where ∂ denotes the (sub-)differential operator.

Hence, we study the special setting where X and Y are both two variable exponent Lebesgue spaces $L^{q(\cdot)}$, that is, Lebesgue spaces where the exponent is not a constant value but rather a function of the position of the domain. We first review the key concept of duality map, providing an explicit formulation, then we apply the iterative scheme to deblurring imaging problems. By using an heuristic adaptive strategy to select the point-wise variable exponent function $q(\cdot)$, numerical tests will show the advantages of considering variable exponent Lebesgue spaces w.r.t. both the standard L^2 Hilbert and the constant exponent Lebesgue space Banach L^q spaces, in terms of both reconstruction quality and

convergence speed.

Keywords: iterative regularization, proximal operators, variable exponent Lebesgue spaces.

References

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