Sobolev orthogonal polynomials and coherent pairs of measures. Computation and applications.

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Abstract

Sequences of monic polynomials orthogonal with respect to a weighted Sobolev inner product

$$(f,g) = \int_{\mathbf{R}} f(x)g(x)d\mu_0(x) + s\int_{\mathbf{R}} f'(x)g'(x)d\mu_1(x), \tag{1}$$

where $s > 0$, and (μ_0, μ_1) are probability measures supported on the real line have received the attention in the last 30 years, see [5] as an updated summary. Computational aspects of these polynomials from a constructive approach as well as the behavior of their zeros have been studied in [3] and [7]. But from an analytic perspective, such polynomials have been considered in some particular cases. In the last years they have been used in the framework of spectral methods for boundary value problems for ordinary linear differential equations, see [2], [8].

In [1] the concept of coherent pairs of probability measures (μ_0, μ_1) supported on the real line is introduced. Indeed, if $\{P_n(x, \mu_i\}_{n\geq 0}, i = 0, 1$, denote the corresponding sequences of orthogonal polynomials, then the following relation holds

$$P_n(x;\mu_1) = \frac{1}{n+1}[P'_{n+1}(x;\mu_0) - \rho_n P'_n(x;\mu_0)], \quad \rho_n \neq 0, \quad n \geq 1. \tag{2}$$

They have been described in [6], where it is proved that one of the measures is the Beta distribution, associated with Jacobi polynomials (resp. Gamma distribution, associated with Laguerre polynomials) and the other one is a rational perturbation of it. Then, you can deduce an useful connection formula between monic Sobolev orthogonal polynomials and those associated withμ_0.

The aim of this lecture is to give an overview on Sobolev orthogonal polynomials. We will focus our attention on constructive methods when you deal with of coherent pairs of measures, Moreover, some analytic properties as well as the computation of their zeros in terms of a generalized eigenvalue problem will be

analyzed. In particular, as an illustrative example, the Laguerre-Sobolev case will be presented, see [4].

Some other families of Sobolev orthogonal polynomials associated for other pairs of measures will be analyzed. Analytic properties of their zeros as well as efficient and computational methods to find them will be described.

Keywords: Weighted Sobolev inner products, coherent pairs of measures, Sobolev orthogonal polynomials, zeros, asymptotic properties, spectral methods for boundary value problems.

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