Error Bounds for Quadrature Rules with Multiple Nodes for Modified Jacobi Weights

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Abstract

In [1], error bounds for Gaussian quadrature formulas were derived for two modifications of Jacobi weight functions on $[-1,1]$:

$$\omega_1(x) = \frac{(1-x)^\alpha(1+x)^\beta}{v-x}, \qquad \omega_2(x) = (v-x)(1-x)^\alpha(1+x)^\beta,$$

with $\alpha, \beta > -1$ and $v > 1$. In the present work, we extend this analysis to quadrature rules with multiple nodes [2, 3] specifically to the $s = 1$ Gauss–Turán-type formula

$$Q_{n,1}[f] = \sum_{\nu=1}^{n} \big[A_\nu\, f(\tau_\nu) + B_\nu\, f'(\tau_\nu) + C_\nu\, f''(\tau_\nu)\big],$$

which achieves degree of exactness $4n-1$ using n nodes with multiplicity three.

For analytic integrands, we obtain error bounds on Bernstein ellipses based on the contour integral representation of the remainder [2]. The per-node convergence rate is $\mathcal{O}(\rho^{-4n})$, doubling the $\mathcal{O}(\rho^{-2n})$ rate of standard Gaussian quadrature [5]. We provide an explicit structural decomposition of the remainder kernel for both weights. Numerical experiments in double and extended precision confirm all theoretical predictions.

Keywords: Gauss–Turán-type quadrature, multiple nodes, error bounds, Bernstein ellipses, modified Jacobi weights

References

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